香 港 大 學
THE UNIVERSITY OF HONG KONG

OOPSLA'24

Imperative Compositional Programming

Type Sound Distributive Intersection Subtyping with References via Bidirectional Typing

Wenjia Ye, Yaozhu Sun, Bruno C. d. S. Oliveira

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Keywords

1. Imperative Compositional Programming
2. Distributive Intersection Subtyping
3. References
4. Type Sound
5. Bidirectional Typing

Compositional Programming

Solving the Expression Problem

Expression

```
type SeqSig<Exp> = {
  Var: String → Exp;
  Ass: String → Exp
              → Exp;
  Seq: Exp → Exp
           → Exp;
}
```

```
type LoopSig<Exp> = {
  WhileDo: Exp → Exp
               → Exp;
  DoWhile: Exp → Exp
               → Exp;
}
```

```
printSeq = trait implements SeqSig<Print> ⇒ {
  (Var     x).print = x;
  (Ass   x e).print = x ++ " = " ++ e.print;
  (Seq e1 e2).print = e1.print ++ "; "
                    ++ e2.print;
}
```

```
printLoop = trait implements LoopSig<Print> ⇒ {
  (WhileDo cond body).print =
      "while (" ++ cond.print ++ ") do { "
                ++ body.print ++ " }";
  (DoWhile body cond).print =
      "do { "  ++ body.print ++ " } while ("
               ++ cond.print ++ ")";
}
```

```
graphSeq = trait
    implements SeqSig<Print ⇒ Graph> ⇒ {
  [self]@(Var x).graph =
      let r = ref (newNode self.print [] [x]) in
      mkPair r r;
  ……
}
```

```
graphLoop = trait implements LoopSig<Graph> ⇒ {
  (WhileDo cond body).graph =
      addAdj (snk cond) [src body] >>
      addAdj (snk cond) [src cond] >>
      mkPair (src cond) (snk cond);
  ……
}
```

```
type Print = { print: String }
```

```
type Graph = { graph: PairRefNode }
```

Dependencies

Operation

Imperative Compositional Programming

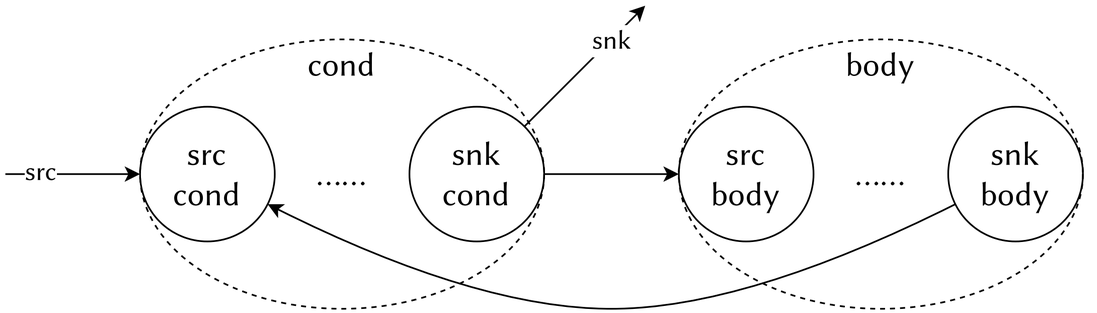
```
type Graph = { graph: PairRefNode };
                    -- (src*,snk*)

type Node = {
  name: String;
  adj: Ref [Ref Node]; -- adjacency array
  ......
};

addAdj (x: Ref Node) (ys: [Ref Node]) =
  !x.adj := !(!x.adj) ++ ys;
```

Dereference

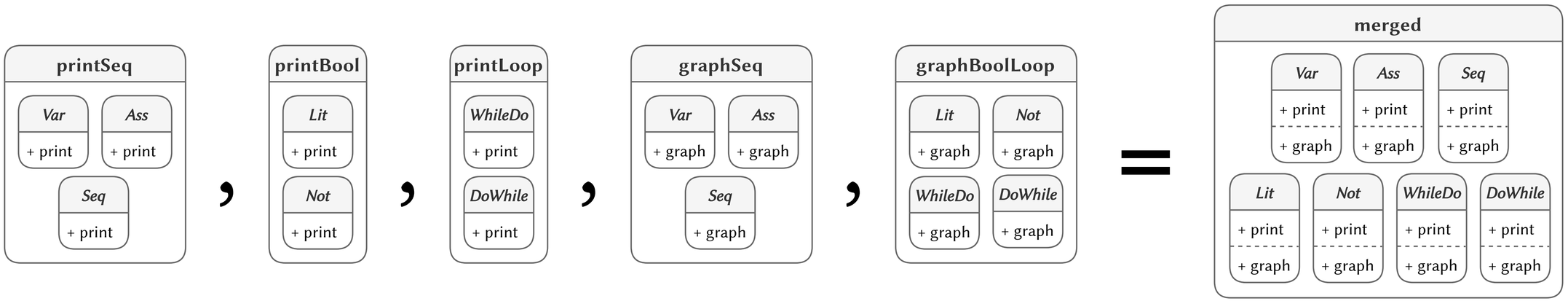
```
graphLoop = trait implements LoopSig<Graph> ⇒ {
  (WhileDo cond body).graph =
      addAdj (snk cond) [src body] >>
      addAdj (snk cond) [src cond] >>
      mkPair (src cond) (snk cond);
  ......
}
```



Nested Trait Composition

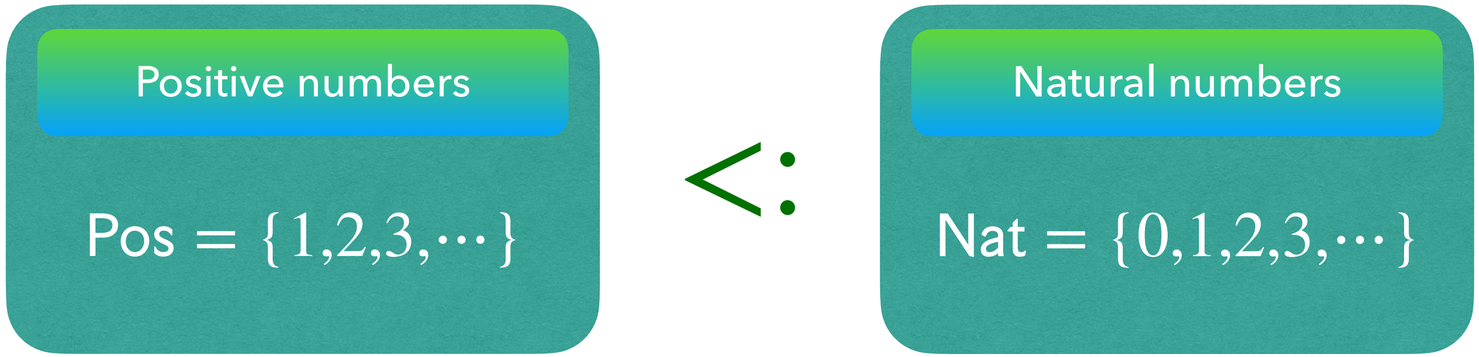
Intersection types

```
merged: SeqSig<Print&Graph> & LoopSig<Print&Graph>
  = new printSeq,graphSeq,printLoop,graphLoop;
```



Intersection Types

$$\frac{e : A \qquad e : B}{e : A \,\&\, B}\ \text{T-Intro}$$



$$\frac{e : A \qquad A <: B}{e : B}\ \text{T-Sub}$$

$$\dfrac{\text{T-Sub}\ \dfrac{1 : \text{Pos} \qquad \text{Pos} <: \text{Nat}}{1 : \text{Nat}} \qquad 1 : \text{Pos}}{1 : \text{Nat} \,\&\, \text{Pos}}\ \text{T-Intro}$$

Intersection Subtyping

$$\frac{}{A \,\&\, B <: A}\ \text{S-AndL} \qquad \frac{}{A \,\&\, B <: B}\ \text{S-AndR} \qquad \frac{A <: B \qquad A <: C}{A <: B \,\&\, C}\ \text{S-And}$$

$$\frac{e : \text{Nat} \,\&\, \text{Pos} \qquad \dfrac{}{\text{Nat} \,\&\, \text{Pos} <: \text{Nat}}\ \text{S-AndL}}{e : \text{Nat}}\ \text{T-Sub} \qquad \frac{e : \text{Nat} \,\&\, \text{Pos} \qquad \dfrac{}{\text{Nat} \,\&\, \text{Pos} <: \text{Pos}}\ \text{S-AndR}}{e : \text{Pos}}\ \text{T-Sub}$$

You can use e as if it simply has type Nat / Pos.

$$\frac{\dfrac{}{\text{Nat} \,\&\, \text{Pos} <: \text{Pos}}\ \text{S-AndR} \qquad \dfrac{}{\text{Nat} \,\&\, \text{Pos} <: \text{Nat}}\ \text{S-AndL}}{\text{Nat} \,\&\, \text{Pos} <: \text{Pos} \,\&\, \text{Nat}}\ \text{S-And}$$

Nat & Pos is equivalent to Pos & Nat.

Reference Types

$$\frac{e : A}{\text{ref } e : \text{Ref } A}\ \text{T-Ref} \qquad \frac{e_1 : \text{Ref } A \qquad e_2 : A}{e_1 := e_2 : ()}\ \text{T-Ass} \qquad \frac{e : \text{Ref } A}{!e : A}\ \text{T-Deref}$$

$$\frac{1 : \text{Pos}}{\text{ref } 1 : \text{Ref Pos}}\ \text{T-Ref}$$

$$\text{T-Sub}\ \frac{x : \text{Ref Pos} \qquad \cancel{\text{Ref Pos} <: \text{Ref Nat}}}{\cancel{x : \text{Ref Nat}}} \Big/ \cancel{(x := 0) : ()}$$

$$\frac{A <: B \qquad B <: A}{\text{Ref } A <: \text{Ref } B}\ \text{S-Ref}$$

[TYPE ERROR]
Subtyping for Ref is invariant!

```
let x = ref 1 : Ref Pos;
x := 0;
!x : Pos
```

Intersection Subtyping with References (1)

$$\frac{e : A}{\mathsf{ref}\ e : \mathsf{Ref}\ A}\ \text{T-Ref} \qquad \frac{e_1 : \mathsf{Ref}\ A \qquad e_2 : A}{e_1 := e_2 : ()}\ \text{T-Ass} \qquad \frac{e : \mathsf{Ref}\ A}{!e : A}\ \text{T-Deref}$$

$$\dfrac{\text{T-Ref}\ \dfrac{1 : \mathsf{Nat}}{\mathsf{ref}\ 1 : \mathsf{Ref}\ \mathsf{Nat}} \qquad \dfrac{1 : \mathsf{Pos}}{\mathsf{ref}\ 1 : \mathsf{Ref}\ \mathsf{Pos}}\ \text{T-Ref}}{\mathsf{ref}\ 1 : \mathsf{Ref}\ \mathsf{Nat} \,\&\, \mathsf{Ref}\ \mathsf{Pos}}\ \text{T-Intro}$$

The definition type-checks!
Initially x points to 1.

```
let x = ref 1 : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Intersection Subtyping with References (2)

$$\frac{e : A}{\mathsf{ref}\ e : \mathsf{Ref}\ A}\ \text{T-Ref} \qquad \frac{e_1 : \mathsf{Ref}\ A \qquad e_2 : A}{e_1 := e_2 : ()}\ \text{T-Ass} \qquad \frac{e : \mathsf{Ref}\ A}{!e : A}\ \text{T-Deref}$$

$$\text{T-Ass}\ \frac{\text{T-Sub}\ \dfrac{x : \mathsf{Ref\ Nat\ \&\ Ref\ Pos} \qquad \mathsf{Ref\ Nat\ \&\ Ref\ Pos} <: \mathsf{Ref\ Nat}}{x : \mathsf{Ref\ Nat}} \qquad 0 : \mathsf{Nat}}{(x := 0) : ()}$$

The assignment type-checks!
Now x points to 0.

```
let x = ref 1 : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Intersection Subtyping with References (3)

$$\frac{e : A}{\mathsf{ref}\ e : \mathsf{Ref}\ A}\ \text{T-Ref} \qquad \frac{e_1 : \mathsf{Ref}\ A \qquad e_2 : A}{e_1 := e_2 : ()}\ \text{T-Ass} \qquad \frac{e : \mathsf{Ref}\ A}{!e : A}\ \text{T-Deref}$$

$$\dfrac{\dfrac{x : \mathsf{Ref\ Nat\ \&\ Ref\ Pos} \qquad \mathsf{Ref\ Nat\ \&\ Ref\ Pos} <: \mathsf{Ref\ Pos}}{x : \mathsf{Ref\ Pos}}\ \text{T-Sub}}{!x : \mathsf{Pos}}\ \text{T-Deref}$$

[RUNTIME ERROR]
0 does not have type Pos!

```
let x = ref 1 : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Type unsound!

Solution by Davies and Pfenning [2000]

$$\frac{e : A \qquad e : B}{e : A \,\&\, B}\ \text{T-Intro}$$

Value restriction

$$\frac{v : A \qquad v : B}{v : A \,\&\, B}\ \text{T-IntroV}$$

$$\frac{\text{T-Ref}\ \dfrac{1 : \text{Nat}}{\text{ref } 1 : \text{Ref Nat}} \qquad \dfrac{1 : \text{Pos}}{\text{ref } 1 : \text{Ref Pos}}\ \text{T-Ref}}{\text{ref } 1 : \text{Ref Nat} \,\&\, \text{Ref Pos}}\ \cancel{\text{T-Intro}}$$

[TYPE ERROR]
"ref 1" is not a value!

```
let x = ref 1 : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Type sound!

Distributive Subtyping Breaks Soundness

$$\frac{}{(A \to B)\ \&\ (A \to C) <: A \to B\ \&\ C}\ \text{S-Dist}$$

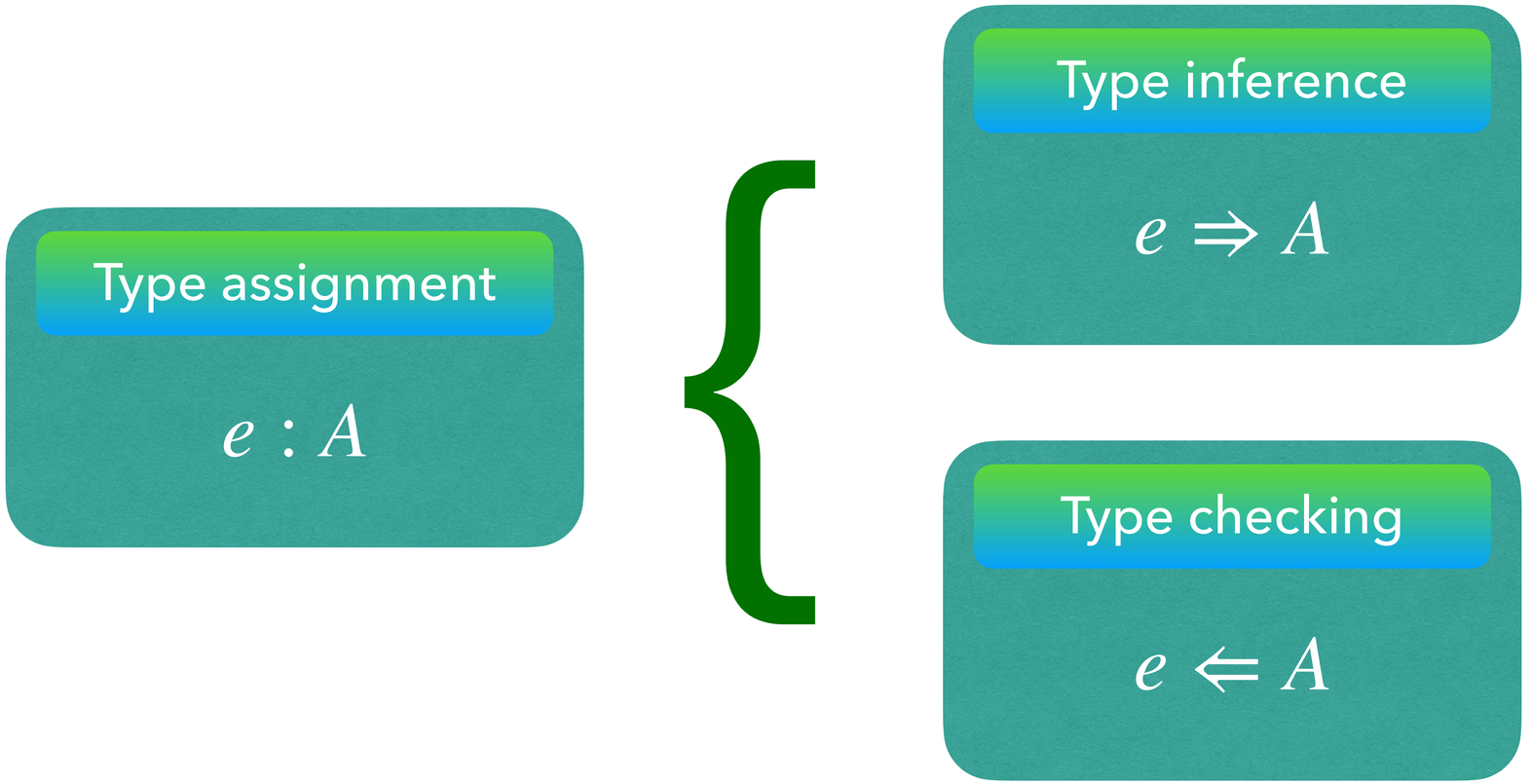
$$\dfrac{\text{T-IntroV}\dfrac{\dfrac{\text{ref } 1 : \text{Ref Nat}}{\lambda x\,.\,\text{ref } 1 : () \to \text{Ref Nat}} \quad \dfrac{\text{ref } 1 : \text{Ref Pos}}{\lambda x\,.\,\text{ref } 1 : () \to \text{Ref Pos}}}{\lambda x\,.\,\text{ref } 1 : (() \to \text{Ref Nat}) \,\&\, (() \to \text{Ref Pos})} \quad \dfrac{}{(() \to \text{Ref Nat}) \,\&\, (() \to \text{Ref Pos}) <: () \to \text{Ref Nat} \,\&\, \text{Ref Pos}}\ \text{S-Dist}}{\lambda x\,.\,\text{ref } 1 : () \to \text{Ref Nat} \,\&\, \text{Ref Pos}}\ \text{T-Sub}$$

The definition type-checks because "λx. ref 1" is a value!

```
let x = (λx. ref 1) () : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Type unsound!

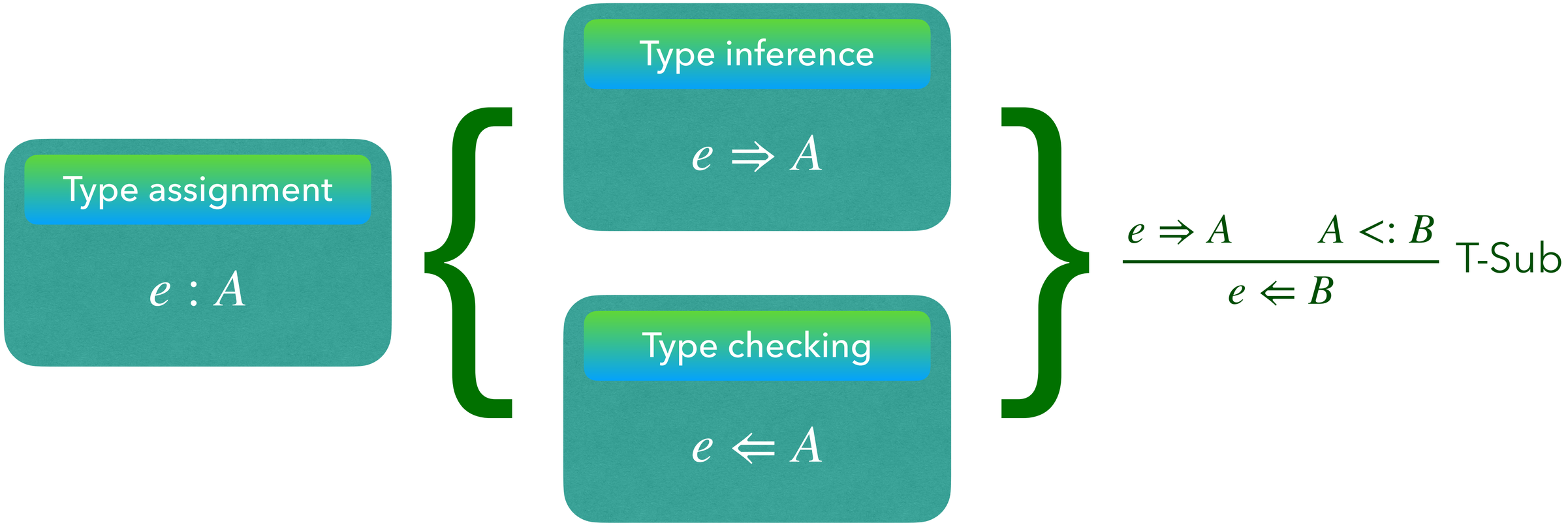
Bidirectional Typing to the Rescue



(usually)
declarative

(usually)
algorithmic

Bidirectional Typing to the Rescue



The bridge between the two typing modes: ***subsumption rule***.

Bidirectional Typing with Numbers

Type assignment (non-bidirectional)

$$\frac{}{1 : \text{Pos}}\ \text{T-Pos} \qquad \frac{1 : \text{Pos} \qquad \text{Pos} <: \text{Nat}}{1 : \text{Nat}}\ \text{T-Sub}$$

Bidirectional typing

$$\frac{}{1 \Rightarrow \text{Pos}}\ \text{T-Pos} \qquad \frac{1 \Rightarrow \text{Pos} \qquad \text{Pos} <: \text{Nat}}{1 \Leftarrow \text{Nat}}\ \text{T-Sub}$$

1 : Pos
✔ infer
✔ check

1 : Nat
✘ infer
✔ check

Bidirectional Typing with References (1)

Type assignment (non-bidirectional)

$$\frac{e : A}{\text{ref } e : \text{Ref } A}\ \text{T-Ref} \qquad \frac{e_1 : \text{Ref } A \quad e_2 \Leftarrow A}{e_1 := e_2 : ()}\ \text{T-Ass} \qquad \frac{e : \text{Ref } A}{!e : A}\ \text{T-Deref}$$

Davies and Pfenning [2000]

$$\frac{e \Leftarrow A}{\text{ref } e \Leftarrow \text{Ref } A}\ \text{T-Ref} \qquad \frac{B \uparrow \text{Ref } A \quad e_1 \Rightarrow B \quad e_2 \Leftarrow A}{e_1 := e_2 : ()}\ \text{T-Ass} \qquad \frac{B \uparrow \text{Ref } A \quad e \Rightarrow B}{!e : A}\ \text{T-Deref}$$

Our approach

$$\frac{e \Rightarrow A}{\text{ref } e \Rightarrow \text{Ref } A}\ \text{T-Ref} \qquad \frac{e_1 \Rightarrow \text{Ref } A \quad e_2 \Leftarrow A}{e_1 := e_2 \Rightarrow ()}\ \text{T-Ass} \qquad \frac{e \Rightarrow \text{Ref } A}{!e \Rightarrow A}\ \text{T-Deref}$$

Bidirectional Typing with References (2)

Our approach

$$\frac{e \Rightarrow A}{\text{ref } e \Rightarrow \text{Ref } A}\ \text{T-Ref}$$

ref 1 : Ref Pos
✔ infer
✔ check

$$\frac{1 \Rightarrow \text{Pos}}{\text{ref } 1 \Rightarrow \text{Ref Pos}}\ \text{T-Ref} \qquad \frac{\text{ref } 1 \Rightarrow \text{Ref Pos} \qquad \text{Ref Pos} <: \text{Ref Pos}}{\text{ref } 1 \Leftarrow \text{Ref Pos}}\ \text{T-Sub}$$

ref 1 : Ref Nat
✘ infer
✘ check

$$\frac{1 \Leftarrow \text{Nat}}{\cancel{\text{ref } 1 \Leftrightarrow \text{Ref Nat}}}\ \text{T-Ref} \qquad \frac{\text{ref } 1 \Rightarrow \text{Ref Pos} \qquad \cancel{\text{Ref Pos} <: \text{Ref Nat}}}{\cancel{\text{ref } 1 \Leftarrow \text{Ref Nat}}}\ \text{T-Sub}$$

Back to the Counterexamples

$$\frac{\cancel{\text{ref } 1 \Leftarrow \text{Ref Nat}} \qquad \text{ref } 1 \Leftarrow \text{Ref Pos}}{\cancel{\text{ref } 1 \Leftarrow \text{Ref Nat} \,\&\, \text{Ref Pos}}}\ \text{T-Intro}$$

$$\frac{\dfrac{\cancel{\text{ref } 1 \Leftarrow \text{Ref Nat}}}{\cancel{\lambda x.\ \text{ref } 1 \Leftarrow () \to \text{Ref Nat}}} \qquad \dfrac{\text{ref } 1 \Leftarrow \text{Ref Pos}}{\lambda x.\ \text{ref } 1 \Leftarrow () \to \text{Ref Pos}}}{\cancel{\lambda x.\ \text{ref } 1 \Leftarrow (() \to \text{Ref Nat}) \,\&\, (() \to \text{Ref Pos})}}\ \text{T-Intro}$$

```
let x = ref 1 : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Type sound!

[TYPE ERROR]
"ref 1" does not have type "Ref Nat"

```
let x = (λx. ref 1) () : Ref Nat & Ref Pos;
x := 0;
!x : Pos
```

Type sound!

Wanna have "Ref Nat" instead of "Ref Pos"?

```
let n = 1 : Nat;
let x = ref n : Ref Nat;
.....
```

Advantages over Davies and Pfenning [2000]

①

$$\frac{e \Leftarrow A \qquad e \Leftarrow B}{e \Leftarrow A \,\&\, B}\ \text{T-Intro}$$

Intersection introduction is standard (w/o value restriction).

②

$$\frac{}{(A \to B) \,\&\, (A \to C) <: A \to B \,\&\, C}\ \text{S-Dist}$$

Distributive subtyping is supported.

Why distributivity matters?

Imperative Compositional Programming (ICP)

- Logical implication is left-distributive over conjunction:

$$A \to B \wedge C \quad \Leftrightarrow \quad (A \to B) \wedge (A \to C)$$

- Distributivity of traits in ICP (encoded as functions of records)

  ↳ nested trait composition [Bi et al. 2018]

  ↳ family polymorphism [Ernst 2001]

  ↳ solution to the expression problem [Wadler 1998]

More in the Paper

- ✦ More introduction to ICP and its type-theoretic foundations.
- ✦ Complete code of modular data-flow analysis in ICP.

- ❖ Formalization of λ_{im}, a calculus similar to that by Davies and Pfenning [2000] but employing our solution based on bidirectional typing.
- ❖ Formalization of F^{+}_{im}, the core calculus for ICP.
- ❖ Mechanized proofs of type soundness for both calculi.

Q&A